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Research Article

Existence of bounded solutions for retarded equations with infinite delay, impulses, and nonlocal condition

Existencia de soluciones limitadas para ecuaciones retardadas con retardo infinito, impulsos y condición no local

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Abstract: In this work, we study the existence of bounded solutions for a semilinear retarded equation with infinite delay, impulse, and non-local conditions. We also show that under some conditions this bounded solution is unique, periodic, or almost periodic depending on the conditions imposed on the terms involving the equation. Through this work, we shall assume that the associated linear equation has an exponential dichotomy, allowing us to find a formula for the bounded solutions, and from this formula, we are able to apply Banach Fixed Point Theorem to prove the existence of such bounded solutions.
Keywords: Exponential dichotomy, bounded solution, unique, periodic, almost periodic.

Resumen: *En este trabajo, estudiamos la existencia de soluciones acotadas para una ecuación retardada semilineal con retardo infinito, impulso y condiciones no locales. También mostramos que bajo algunas condiciones esta solución acotada es única, periódica o casi periódica dependiendo de las condiciones impuestas a los términos que involucran la ecuación. A través de este trabajo, asumiremos que la ecuación lineal asociada tiene una dicotomía exponencial, lo que nos permite encontrar una fórmula para las soluciones acotadas, y a partir de esta fórmula, podemos aplicar el Teorema del punto fijo de Banach para demostrar la existencia de tales soluciones acotadas.*
Palabras clave: *Cuasi periódica, dicotomía exponencial, periódica, solución acotada, única.*

1 Introduction

There are many works on the existence of bounded solutions without impulses, non-local conditions and delay simultaneously, to mention we have the works done in((Leiva, 1999a, 2000, 1999b; Leiva & Sivoli, 2003; Leiva & Sequera, 2003; Leiva & Sivoli, 2018; Liu, 2000; Liu *et al.*, 2006)). Recently, in (Ayala *et al.*, 2020) the existence of solutions for retarded equations with infinite delay, impulses, and non-local conditions has been proved using Karakosta's fixed point theorem. In (Abbas *et al.*, 2020), the existence of periodic mild solutions of infinite delay evolution equations with non-instantaneous impulses has been studied, by using Poincare map, measure of non-compactness and Darbo fixed point theorem. Compared with these works, in addition we have non-local conditions, and first, we prove the existence of bounded solutions, and under son conditions these bounded solutions are stable, periodic or almost periodic depending on the conditions impose to the linear and non-linear term. Without further ado, in this work we shall study the existence of bounded solutions for the following semi-linear non-autonomous retarded equation with infinite delay, impulses and non-local condition:

$$\begin{cases} z' = A(t)z + f(t,z_t), & t > 0, t \neq t_k, \\ z(s) + g(z_{\tau_1}, \cdots, z_{\tau_q})(s) = \phi(s), & s \in (-\infty, 0) = \mathbb{R}_-, \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), & k = 1, 2, \cdots, P, \end{cases}$$

where $A(t)$ is a continuous $n \times n$ matrix defined on $\mathbb{R}$, $\phi \in \mathcal{PW}$ the space defined as follows

$\mathcal{PW} = \{\phi : (-\infty, 0] \to \mathbb{R}^n : \phi$ is bounded and continuous except in a finite number of point, $s_{\phi k}, k = 1, 2, \ldots, p,$ where the side limits exists $\phi(s_{\phi k}^-)$, $\phi(s_{\phi k}^+) = \phi(s_{\phi k})\}$

endowed with the norm

$$\| \phi \|_{\mathcal{PW}} = \sup_{s \in \mathbb{R}_-} \| \phi(s) \|,$$

where the side limits are defined as follows $\phi(s_{\phi k}^+) = \lim_{s \to s_{\phi k}^+} \phi(s)$ and $\phi(s_{\phi k}^-) = \lim_{s \to s_{\phi k}^-} \phi(s)$.

Here, $0 < t_1 < t_2 < \cdots < t_p$, $0 < \tau_1 < \tau_2 < \ldots < \tau_q$, and the functions

$$g : (\mathcal{PW})^q \to \mathcal{PW}, \quad J_K : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad f : \mathbb{R} \times \mathcal{PW} \to \mathbb{R}^n$$

are smooth enough such that the problem (1) admits only one solution $z(t)$ (see (Ayala *et al.*, 2020)) given by

$$\begin{aligned} z(t) = \; & U(t,0)[\phi(0) - g\left(z_{\tau_1}, \cdots, z_{\tau_q}\right)(0)] \\ & + \int_0^t U(t,s) f\left(s, z_s\right) ds + \Sigma_{0 < t_k < t} U\left(t, t_k\right) J_k\left(t_k, z\left(t_k\right)\right), \quad t \in [0, \tau] \\ z(s) = \; & \phi(s) - g(z_{\tau_1}, \cdots, z_{\tau_q})(s), \quad s \in \mathbb{R}_-. \end{aligned}$$

The space $(\mathcal{PW})^q$ is endowed with the usual norm, and $U(t,s) = \Phi(t)\Phi^{-1}(s)$, where $\Phi(\cdot)$ is the fundamental matrix of the linear system

$$z'(t) = A(t)z(t), t \in \mathbb{R} \tag{1}$$

i.e.,

$$\begin{cases} \Phi'(t) = A(t)\Phi(t) \\ \Phi(0) = I \end{cases}$$

2 Preliminaries

In this section, we shall choose the space where this problem will be set. To this end, we shall define the following Banach space:

$\mathcal{PW}_b(\mathbb{R}; \mathbb{R}^n) = \{z : \mathbb{R} \to \mathbb{R}^n : | .z |_{\mathbb{R}_-} \in \mathcal{PW}$ and $z\big|_{\mathbb{R}_+}$ is bounded and continuous except at the point t_k, $k = 1, 2, \ldots, p,$ where $z(t_k^+), z(t_k^-)$ exist and $z(t_k^+) = z(t_k)\}$,

endowed with the norm

$$\| z \|_b = \sup_{t \in \mathbb{R}} \| z(t) \|, \quad z \in \mathcal{PW}_b.$$

Now, we shall assume the following hypotheses:

H1) The linear system (1) admits an exponential dichotomy on $\mathbb{R}$. That is to say, there exist a continuous projection $P(t)$, $t \in \mathbb{R}$, $\beta > 0$ and $M \geq 1$ such that

i) $U(t,s)P(s) = P(t)U(t,s), \quad t, s \in \mathbb{R},$

ii) $\| U(t,s)(I - P(s)) \| \leqslant Me^{\beta(t-s)}, \quad t \geqslant s,$

iii) $\| U(t,s)P(s) \| \leqslant Me^{\beta(t-s)}, \quad t \leqslant s$

H2) There exists $\gamma > 0$ and $\ell > 0$ such that

$$\| g\left(z_1, z_2, \ldots z_q\right) \|_{\mathcal{PW}} < \frac{\ell}{6}, \quad z \in \mathcal{PW},$$

$$\| g\left(z_1, \ldots, z_q\right) - g\left(w_1, \cdots, w_q\right) \|_{\mathcal{PW}} < \gamma \sup_{t \geqslant a} \|z(t) - w(t)\|_q$$

with

$$\| z(t)-w(t) \|_q := \sum_{i=1}^{q} \| z_i(t)-w_i(t) \|_{\mathbb{R}^n} .$$

H3) The function f satisfies the following local Lipschitz condition:
Given an interval $[a,b]$ and a ball $\mathcal{B}_\gamma(0) \subset \mathcal{PW}$, there exists a constant $\mathcal{K}>0$ such that

$$\| f(t,z_1)-f(t,z_2) \|_{\mathbb{R}^n} \leqslant \mathcal{K}|t-s| + \| z_1-z_2 \|_{\mathcal{PW}}, \quad z_1,z_2 \in \mathcal{B}_\gamma(0), \quad t,s, \in [a,b].$$

Also, there exists a constant $L_f>0$ such that

$$\| f(t,0) \|_{\mathbb{R}^n} \leqslant L_f, \quad t \in \mathbb{R}.$$

H4) There are constants $S_k, L_k > 0, \quad k=1,2,\cdots,p$, such that

$$\| J_K(t,z_1)-J_K(t,z_2) \|_{\mathbb{R}^n} \leqslant S_k \| z_1-z_2 \|_{\mathbb{R}^n}, \quad \forall z_1,z_2 \in \mathbb{R}^n, \quad \forall t \in \mathbb{R};$$

and

$$\| J_K(t,0) \|_{\mathbb{R}^n} < L_k, \quad k=1,2,\cdots,p, \quad t \in \mathbb{R}.$$

Lemma 1. *Under the hypotheses **H1)-H4)**. A function z belonging to $\mathcal{PW}_b$ is a solution of (1) if, and only if, z is a solution of the following integral equation*

$$\begin{aligned} z(t) &= \int_{-\infty}^{\infty} G(t,s) f(s,z_s)\,ds + \sum_{0<t_k \leqslant t_p} G(t,t_k) J_k(t_k, z(t_k)), \qquad (2)\\ z(s) &= \phi(s) - g(z_{\tau_1},\cdots,z_{\tau_q})(s), \quad s \in (-\infty,0] = \mathbb{R}_-, \end{aligned}$$

where $G(t,s)$ is the Green function defined by

$$G(t,s) = \begin{cases} U(t,s)(I-P(s)), & t \geqslant s, \\ -U(t,s)P(s), & t \leqslant s. \end{cases} \qquad (3)$$

Proof. Suppose that for some $\rho>0, \quad z \in \mathcal{B}_\rho^b(0) \subset \mathcal{PW}_b$, where $\mathcal{B}_\rho^b(0)$ is the ball of center zero and radius $\rho>0$ in $\mathcal{PW}_b$, i.e.,

$$\mathcal{B}_\rho^b(0) = \{ z \in \mathcal{PW}_b : \| z \|_b < \rho \}.$$

Let L_ρ be the Lipschitz constant of f in $\mathcal{B}_\rho^b(0)$. On the other hand, we have that

$$\begin{aligned} z(t) &= U(t,0)\left[\phi(0) - g\left(z_{\tau_1},\cdots,z_{\tau_q}\right)(0)\right] \\ &\quad + \int_0^t U(t,s) f(s,z_s)\,ds + \sum_{0<t_k<t} U(t,t_k) J_k(t_k,z(t_k)) \\ &= U(t,t_0)\Bigg\{ U(t_0,0)\left[\phi(0) - g\left(z_{\tau_1},\cdots,z_{\tau_q}\right)(0)\right] \\ &\quad + \int_0^{t_0} U(t_0,s) f(s,z_s)\,ds + \sum_{0<t_k<t_0} U(t_0,t_k) J_k(t_k,z(t_k)) \Bigg\}, \\ &\quad + \int_{t_0}^t U(t,s) f(s,z_s)\,ds + \sum_{t_0<t_k<t} U(t,t_k) J_k(t_k,z(t_k)), \end{aligned}$$

for $t_0 < t$. So,

$$z(t) = U(t,t_0) z(t_0) + \int_{t_0}^t U(t,s) f(s,z_s)\,ds + \sum_{t_0<t_k<t} U(t,t_k) J_k(t_k,z(t_k)).$$

Hence,

$$\begin{aligned}(I-P(t))z(t) &= (I-P(t))U(t,t_0)z(t_0)+\int_{t_0}^{t}(I-P(t))U(t,s)f(s,z_s)\,ds\\ &+\sum_{t_0<t_k<t}(I-P(t))U(t,t_k)J_k(t_k,z(t_k))\\ &= U(t,t_0)(I-P(t_0))z(t_0)+\int_{t_0}^{t}U(t,s)(I-P(s))f(s,z_s)\,ds\\ &+\sum_{t_0<t_k<t}U(t,t_k)(I-P(t_k))J_K(t_k,z(t_k)).\end{aligned}$$

On the other hand,

$$\| U(t,t_0)(I-P(t_0))z(t_0)\| \leqslant M\| z(t_0)\| e^{-\beta(t-t_0)}, \quad t_0\leqslant t.$$

But, $\| z\|_b<\rho$. Then,

$$\| U(t,t_0)(I-P(t_0))z(t_0)\| \leqslant M\rho e^{-\beta(t-t_0)}.$$

Passing to the limit as $t_0\to-\infty$, we obtain that

$$\lim_{t_0\to-\infty}\| U(t,t_0)(I-P(t_0))z(t_0)\| = 0.$$

Therefore, we get that

$$(I-P(t))z(t)=\int_{-\infty}^{t}U(t,s)(I-P(s))f(s,z)ds+\sum_{0<t_k<t.}(U(t,t_k)(I-P(t_k))J_k(t_k,z(t_k))). \tag{4}$$

Now, let us prove that this improper integral converges.

$$\begin{aligned}\| \int_{-\infty}^{t}U(t,s)(I-P(s))f(s,z_s)ds\| &\leqslant \int_{-\infty}^{t}\| U(t,s)(I-P(s))f(s,z_s)ds\\ &\leqslant \int_{-\infty}^{t}Me^{-\beta(t-s)}\| f(s,z_s)\| ds\\ &= \int_{-\infty}^{t}Me^{-\beta(t-s)}\| f(s,z_s)-f(s,0)+f(s,0)\| ds\\ &\leqslant \int_{-\infty}^{t}Me^{-\beta(t-s)}(L_\rho\rho\| z_s\| +L_f)ds\\ &= \frac{M(L_\rho\rho+L_f)}{\beta}<\infty.\end{aligned}$$

Now, we shall suppose that $t_0>t$. Then,

$$
\begin{aligned}
P(t)z(t) &= P(t)U(t,0)\left[\phi(0) - g(t(z_{\tau_1}, z_{\tau_2}, \cdots, z_{\tau_q})(0)\right] \\
&\quad + \int_0^t P(t)U(t,s) f(s, z_s)\, ds + \sum_{0<t_k<t} P(t)U(t,t_k) J_k(t_k, z(t_k)) \\
&= P(t)U(t,t_0 \Big\{ U(t_0,0)\left[\phi(0) - g\left(z_{\tau_1}, z_{\tau_2}, \cdots, z_{\tau_q}\right)(0)\right] \\
&\quad + \int_0^{t_0} U(t_0,s) f(s, z_s)\, ds + \sum_{0<t_k<t_0} U(t_0,t_k) J_k(t_k, z(t_k)) \Big\} \\
&\quad + \int_{t_0}^t U(t,s)P(s) f(s, z_s)\, ds + \sum_{0<t_k<t} U(t,t_k) P(t_k) J_k(t_k, z(t_k)) \\
&\quad - P(t)U(t,t_0) \sum_{0<t_k<t_0} U(t_0,t_k) J_k(t_k, z(t_k)) \\
&= P(t)U(t,t_0) z(t_0) + \int_{t_0}^t U(t,s)P(s) f(s, z_s) ds \\
&\quad + \sum_{0<t_k<t} U(t,t_k) P(t_k) J(t_k, z(t_k)) - \sum_{t<t_k<t_0} U(t,t_k) P(t_k) J_k(t_k, z(t_k)) \\
&= U(t,t_0) P(t_0) z(t_0) + \int_{t_0}^t U(t,s)P(s) f(s, z_s)\, ds \\
&\quad - \sum_{t<t_k<t_0} U(t,t_k) P(t_k) J_K(t_k, z(t_k)).
\end{aligned}
$$

From hypothesis **H1**$) - iii)$, we get that

$$
\lim_{t_0 \to +\infty} \| U(t,t_0) P(t_0) z(t_0) \| = 0.
$$

Hence,

$$
P(t)z(t) = -\int_t^\infty U(t,s)P(s) f(s, z_s)\, ds - \sum_{t<t_k \leqslant t_p} U(t,t_k) P(t_k) J_k(t_k, z(t_k)).
$$

Let us prove that this improper integral converges.

$$
\begin{aligned}
\| -\int_t^\infty U(t,s)P(s) f(s, z_s)\, ds \| &\leqslant \int_t^\infty \| U(t,s)P(s) f(s, z_s) \|\, ds \\
&\leqslant \int_t^\infty M e^{\beta(t-s)} \| f(s, z_s) - f(s,0) + f(s,0) \|\, ds \\
&\leqslant \int_t^\infty M e^{\beta(t-s)} \left(L_\rho \| z_s \| + L_f\right) ds \\
&\leqslant M\left(L_\rho \rho + L_f\right) \int_t^\infty e^{\beta(t-s)} ds \\
&= M\left(L_\rho \rho + L_f\right) \left[\frac{e^{\beta(t-s)}}{-\beta}\right]_t^\infty \\
&= \frac{M\left(L_\rho \rho + L_f\right)}{\beta} < \infty.
\end{aligned}
$$

On the other hand,

$$\begin{aligned} z(t) &= (I-P(t))z(t)+P(t)z(t) \\ &= \int_{-\infty}^{\infty} G(t,s)f(s,z_s)\,ds + \sum_{0<t_k\leqslant t_p} G(t,t_k)\,J_k(t_k,z(t_k)). \end{aligned}$$

Now, suppose that z is a solution of the integral equation (2). Then,

$$\begin{aligned} z(t) = &\int_{-\infty}^{t} U(t,s)(I-P(s))f(s,z_s)\,ds - \int_{t}^{\infty} U(t,s)P(s)f(s,z_s)\,ds \\ &+ \sum_{0<t_k<t} U(t,t_k)(I-P(t_k)\,J_k(t_k,z(t_k)) - \sum_{t<t_k<t_p} U(t,t_k)\,P(t_k)\,J_K(t_k,z(t_k)). \end{aligned}$$

Therefore,

$$\begin{aligned} z'(t) = &\int_{-\infty}^{t} A(t)U(t,s)(I-P(s))f(s,z_s)\,ds - \int_{t}^{\infty} A(t)U(t,s)P(s)f(s,z_s)\,ds \\ &+ (I-P(t))\,f(t,z_t) + P(t)f(t,z_t) + \sum_{0<t_k<t} A(t)U(t,t_k)(I-P(t_k))\,J_k(t_k,z(t_k)) \\ &- \sum_{t<t_k<t_p} A(t)U(t,t_k)(P(t_k))\,J_k(t_k,z(t_k)). \end{aligned}$$

Hence,

$$z'(t) = A(t)z(t) + f(t,z_t), \quad t \geqslant 0.$$

□

3 Existence of bounded solutions

In this section, we shall prove the existence of bounded solutions for the system (1), and under some conditions, we prove the uniqueness of such a bounded solution. Also, under additional conditions, we prove the stability of these bounded solutions as well.

Theorem 1. *Assume the hypotheses **H1)-H4)**. Let $\mathcal{B}_\rho^b$ be the ball of center zero and radius ρ in $\mathcal{PW}$, and L_ρ the Lipschitz constant of f in $\mathcal{B}_{2\rho}$. If the following estimate holds*

$$\rho\left(1 - M\left(\frac{2L_\rho + \beta S}{\beta}\right)\right) > M\left(\frac{2L_f + \beta\widetilde{L}}{\beta}\right) \tag{5}$$

where $S = \sum_{0<t_k<t} s_k$ and $\widetilde{L} = \sum_{K=1}^{P} L_k$, then the system (1) admits one, and only one, bounded solution z^b with $\| z^b(t) \| \leqslant \rho, \quad t \in \mathbb{R}$. Moreover, if additionally we assume that $P(t) \equiv 0$ and

$$\frac{5}{6} + \frac{3ML\rho}{\beta} + 3SM < 1 \text{ and } \ell \leqslant S \tag{6}$$

this bounded solution is locally stable.

Proof. From Lemma 1, it is enough to prove that the operator

$$\begin{array}{llll} \mathcal{K}: & \mathcal{PW}_b & \longrightarrow & \mathcal{PW}_b \\ & t & \longmapsto & (\mathcal{K}z)(t) = \int_{-\infty}^{\infty} G(t,s)f(s,z_s)\,ds + \sum_{0<t_k\leqslant t_p} G(t,t_k)\,J_k(t_k,z(t_k)), \end{array}$$

has a fixed point in $\mathcal{B}_\rho^b$. For $z \in \mathcal{B}_\rho^b$, we have the following estimate

$$\| (\mathcal{K}z)(t) \|_{\mathbb{R}^n} \leq \int_{-\infty}^{\infty} \| G(t,s) \| \| f(s,z_s) \| ds + \sum_{0<t_k\leqslant t_p} \| G(t,t_k) \| J_k(t_k, z(t_k)) \| .$$

From the definition of the Green function, we obtain that

$$\| G(t,s) \| \leqslant Me^{-\beta|t-s|}, \quad t,s \in \mathbb{R}.$$

Therefore,

$$\begin{aligned}
\| (\mathcal{K}z(t) \| \leqslant & \int_{-\infty}^{\infty} Me^{-\beta|t-s|} \| f(s,z_s) - f(s,0) + f(s,0) \| ds \\
& + \sum_{0<t_k<t_p} Me^{-\beta|t-t_k|} \| J_k(t_k, z(t_k)) - J_k(t_k,0) - J_k(t_k,0) \| \\
& \leqslant \int_{-\infty}^{\infty} e^{(\beta|t-s|} \{ L_\rho \| z_s \| + L_f \} ds + \sum_{k=1}^{P} M \{ S_k \| z(t_k) \| + L_k \} \\
& \leqslant M \{ L_\rho \rho + L_f \} \left(\int_t^{\infty} e^{\beta(t-s)} + \int_{-\infty}^{t} e^{-\beta(t-s)} \right) + M\rho \sum_{k=1}^{P} s_k + M \sum_{k=1}^{P} L_k \\
& \leqslant M\{L_\rho \rho + L\rho\} \left(\frac{1}{\beta} + \frac{1}{\beta} \right) + M_\rho S + M\widetilde{L} \\
& = \frac{2M\{L_\rho \rho + L_f\}}{\beta} + M\{\rho S + \widetilde{L}\}.
\end{aligned}$$

From (5), we get that

$$\| (\mathcal{K}z)(t) \| < \rho \implies \| \mathcal{K}z \|_b < \rho \iff \mathcal{K}(\mathcal{B}) \subset \mathcal{B}_\rho^b.$$

$$\begin{aligned}
\| ((\mathcal{K}z)(t) - (\mathcal{K}\tilde{z})(t) \| \| \leqslant & \int_{-\infty}^{\infty} Me^{\beta|t-s|} \| (f,z_s) - f(s,\widetilde{z}_s) \|_{ds} \\
& + \sum_{k=1}^{P} M \| J_k(t_{k,}, z(t_k)) - J_k(t_k, \tilde{z}(t_k)) \| \\
& \leqslant ML_\rho \| z - \widetilde{z} \| \int_{\infty}^{\infty} e^{-\beta|t-s|} ds \\
& + M \sum_{k=1}^{P} S_k \| z(t_k) - \widetilde{z}(t_k) \| \\
& \leqslant \frac{2ML_\rho}{\beta} \| z - \widetilde{z} \| + MS \| z - \widetilde{z} \| \\
& = \left(\frac{2ML_\rho + \beta MS}{\beta} \right) \| z - \widetilde{z} \| \\
& = M \left(\frac{2L_\rho + \beta S}{\beta} \right) \| z - \widetilde{z} \| .
\end{aligned}$$

From (5), we know that

$$M \left(\frac{2L_\rho + \beta S}{\beta} \right) < 1,$$

which implies that $\mathcal{K}$ is a contraction. Then, applying Banach fixed point Theorem, we get that $\mathcal{K}$ has a unique fixed point in the ball $\mathcal{B}_\rho^b$, i.e., there exists $z^b \in \mathcal{B}_\rho^b$, such that

$$z^b = \mathcal{K}z^b.$$

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Hence,

$$z^b(t)=\int_{-\infty}^{\infty}G(t,s)f(s,z_s^b)ds+\sum_{0<t_k\leqslant t_p}G(t,t_k)J\left(t_k,z^b(t_k)\right).$$

To prove that $z^b(t)$ is locally stable, we consider any other solution $z(t)$ of (1) such that $\left\|z(t_0)-z^b(t_0)\right\|<\rho/2$. Then $\| z(t_0)<2\rho$. As long as $\|z(t)\|$ remains less than 2ρ, we get the following estimate

$$\begin{aligned}\| z(t)-z^b(t)\| \leqslant \| U(t,t_0)\| \left[\| z(t_0)-z^b(t_0)\| + \| g(z_{\tau_1},\dots{}_{\tau_q})(t_0)-g\left(z^b_{\tau_1},\dots,z^b_{\tau_q}\right)(t_0)\|\right]\\ \int_{t_0}^{t}\| U(t,s)\| \| f(s,z_s)-f\left(s,z_s^b\right)\| ds\\ +\sum_{0<t_k<t}\| U(t,t_k)\| \| J_K(t_k,z(t_k))-J_k(t_k,z(t_k))\| .\end{aligned}$$

Since $P(t)\equiv 0$, then

$$\| U(t,s)\| \leqslant Me^{-\beta(t-s)},\quad t\geqslant s.$$

Therefore,

$$\begin{aligned}\| z(t)-z^b(t)\| \leqslant M\| z(t)-z^b(t)\| + \| g(z_{\tau_1},\dots{}_{\tau_q})-g\left(z^b_{\tau_1},\dots,z^b_{\tau_q}\right)\|\\ +M\int_{t_0}^{t}e^{-\beta(t-s)}\| f(s,z_s)-f\left(s,z_s^b\right)\| ds\\ +\sum_{0<t_k<t}Me^{-\beta(t-t_k)}\| J_k(t_k,z(t_k))-J_k\left(t_k,z^b(t_k)\right)\| .\end{aligned}$$

Let $t_1=\sup\{t>t_0:\|z(t)\|<2\rho\}$. Then either $t_1=\infty$ or $\|z(t_1)\|=2\rho$. Then, from the above estimate one can get that

$$\begin{aligned}\| z(t)-z^b(t)\| &< \frac{\rho}{2}+\frac{2\ell}{6}+\frac{M}{\beta}L_\rho \sup_{S\in[t_0,t_1]}\| z(s)-z^b(s)\| +M\sum_{0<t_k<t_1}S_k\| z(t_k)-z^b(t_k)\|\\ &<\frac{\rho}{2}+\frac{2\ell}{6}+3\rho\frac{M}{\beta}L_\rho+3\rho MS\\ &<\left(\frac{1}{2}+\frac{1}{3}+\frac{3ML_\rho}{\beta}+3MS\right)\rho\\ &=\left(\frac{5}{6}+\frac{3ML_\rho}{\beta}+3S\right)\rho.\end{aligned}$$

Thus,

$$\rho<\left(\frac{5}{6}+\frac{3ML_\rho}{\beta}+3MS\right)\rho,$$

which is a contradiction. Therefore, $t_1=\infty$ and $z(t)\in\mathcal{B}^b_{2\rho},\quad t\geqslant t_0$. Now, define

$$\| z-z^b\|_+=\sup_{t\geqslant t_0}\| z(t)-z^b(t)\|$$

Then,

$$\begin{aligned}\| z-z^b\|_+ &\leqslant \| z(t_0)-z^b(t_0)\| + \| g(z_{\tau_1},\dots{}_{\tau_q})-g\left(z^b_{\tau_1},\dots,z^b_{\tau_q}\right)\| +\frac{L_\rho M}{\beta}\| z-z^b\|_+ +SM\| z-z^b\|_+\\ &\leqslant \| z(t_0)-z^b(t_0)\| +\gamma\| z(\cdot)-z^b{}_+ +\frac{M^{L\rho}}{\beta}\| z-z^b\|_+ +SM\| z-z^b\|_+\end{aligned}$$

which implies that

$$\left(1-\Theta-\frac{ML_\rho}{\beta}-MS\right)\| z-z_0\| \leqslant \| z(t_0)-z^b(t_0)\|$$

By putting $\Theta = \gamma + \frac{ML_\rho}{\beta} + MS$, we obtain that

$$\| z - z_1^b \|_+ \leqslant \frac{1}{1-\Theta} \| z(t_0) - z^b(t_0) \| .$$

This implies the stability. □

To prove the uniqueness of the bounded solution globally, we need the following additional hypothesis:
H5) The function f is globally Lipschitz, i.e., there exists a constant $L > 0$ such that

$$\| f(t,z_1) - f(s,z_2) \| < L\{|t-s| + \| z_1 - z_2 \|_{\mathcal{PW}}\} \quad \forall t,s, \in \mathbb{R}, \quad \forall z_1, z_2 \in \mathcal{PW}.$$

Theorem 2. *Suppose the hypotheses **H1),H2),H3),H5** hold and*

$$0 < \frac{ML}{\beta} + SM < \frac{1}{6}.$$

Then the equation (1) admits one, and only one, bounded solution $z^b(t)$ for $t \in \mathbb{R}$. Moreover, if condition (6) holds, then this bounded solution is globally uniformly stable.

Proof. Let $L > 0$ be the Lipschitz constant of f. Then, there exists $\rho_1 > 0$ such that

$$\left(1 - \frac{6ML}{\beta} - 6MS\right)\rho_1 > \frac{ML}{\beta} + \widetilde{L}M$$

$$\Longleftrightarrow \left(1 - \frac{6ML - 6\beta MS}{\beta}\right)\rho_1 > \frac{ML + \beta\widetilde{L}M}{\beta}.$$

Then, applying Theorem 1, for each $\rho > \rho_1$, we obtain the existence of an unique bounded solution of system (1) in the ball $\mathcal{B}_\rho^b$. Hence the problem (1) has one, and only one, globally bounded solution z^b. To prove the uniform stability, we assume that $P(t) \equiv 0$, consider other solution $z(t)$ of (1), and the following estimate

$$\| z - z^b \|_+ \leqslant \frac{1}{1-\Theta} \| z(t_0) - z^b(t_0) \|,$$

where

$$\Theta = \left[\gamma + \frac{ML}{\beta} + MS\right].$$

Since Θ does not depend on ρ and t_0, the stability is globally uniform. □

4 Periodic and Almost periodic solutions

In this section, we shall prove that under some additional conditions, the bounded solutions give by Theorems 1 and 2 are periodic or almost periodic. To this end, in order to prove the periodicity of the bounded solution $z^b(\cdot)$, we shall assume the following hypotheses:
H6) $f(t,\phi) = f(t+T,\phi), \quad t \in \mathbb{R}, \quad \phi \in \mathcal{PW}.$
H7) $A(t+T) = A(t), \quad t \in \mathbb{R}.$
From the Floquet Theory, there exists a continuous periodic matrix $D(t)$ and a constant matrix L such that for $t \in \mathbb{R}$

$$D(t+T) = D(t), \quad \text{and} \quad \Phi(t) = D(t)e^{Lt}.$$

From here we get that

$$U(t+T,s+T) = D(t+T)e^{L(t+T)}e^{-L(s+T)}D^{-1}(s+T) = D(t)e^{L(t-s)}D^{-1}(s) = U(t,s).$$

Lemma 2. *Under the hypotheses **H6**)-**H7**) the unique bounded solution $z^b(\cdot)$ given in Theorem 1 and Theorem 2 is also T-periodic for $t > t_p$.*

Proof. Let z^b be the unique solution of (1) in the ball $\mathcal{B}^b_\rho$. Now, we shall prove that $z(t) = z^b(t+T)_b$ is also a solution of (1) in the ball $\mathcal{B}^b_\rho$ for $t > t_0 > t_p$. Observe that

$$z^b_{s+T}(u) = z^b(s+u+T) = z(s+u) = z_u(s).$$

Let $t > t_0 > t_p$, and consider

$$\begin{aligned}
z^b(t) &= \int_{-\infty}^{\infty} G(t,s)f(s,z^b_s)ds + \sum_{0<t_k\leqslant t_p} G(t,t_k)J(t_k,z^b(t_k)) \\
&= \int_{-\infty}^{t} U(t,s)(I-P(s))f(s,z^b_s)ds - \int_{t}^{\infty} U(t,s)P(s)f(s,z^b_s)ds \\
&+ \sum_{0<t_k\leqslant t_p} U(t,t_k)(I-P(t_k))J_k(t_k,z^b(t_k)) - \sum_{t<t_k\leqslant t_p} U(t,t_k)P(t_k)J_k(t_k,z^b(t_k)) \\
&= \int_{-\infty}^{t_0} U(t,s)(I-P(s))f(s,z^b_s)ds + \int_{t_0}^{t} U(t,s)(I-P(s))f(s,z^b_s)ds \\
&- \left[\int_{t_0}^{t} U(t,s)P(s)f(s,z_s)\,ds + \int_{-\infty}^{t_0} U(t,s)P(s)f(s,z_s)\,ds\right] \\
&+ \sum_{0<t_k\leqslant t_p} U(t,t_k)(I-P(t_k))J_k(t_k,z^b(t_k)) \\
&= \int_{-\infty}^{\infty} G(t,s)(I-P(s))f\left(s,z^b_s\right)ds \\
&+ \int_{t_0}^{t} G(t,s)f\left(s,z^b_s\right)ds + \sum_{0<t_k\leqslant t_p} U(t,t_k)(I-P(t_k))J_k(t_k,z^b(t_k)) \\
&= U(t,t_0)[\int_{-\infty}^{\infty} G(t_0,s)f(s,z^b_s)ds + \sum_{0<t_k\leqslant t_p} U(t_0,t_k)(I-P(t_k))J_k(t_k,z^b(t_k))] \\
&+ \int_{t_0}^{t} G(t,s)f(s,z^b_s)ds.
\end{aligned}$$

Therefore, for $t > t_0 > t_p$, we have that

$$z^b(t) = U(t,t_0)z^b(t_0) + \int_{t_0}^{t} G(t,s)f\left(s,z^b_s\right)ds. \tag{7}$$

Hence,

$$\begin{aligned}
z^b(t+T) &= U(t+T,t_0)z^b(t_0) + \int_{t_0}^{t+T} G(t+T,s)f\left(s,z^b_s\right)ds \\
&= U(t+T,t_0)z^b(t_0) + \int_{t_0-T}^{t} G(t+T,s+T)f\left(s+T,z^b_{s+T}\right)ds \\
&= U(t+T,t_0+T)U(t_0+T,t_0)z^b(t_0) + \int_{t_0-T}^{t} G(t,s)f(s,z_s)\,ds \\
&= U(t,t_0)U(t_0+T,t_0)z^b(t_0) + \int_{t_0-T}^{t_0} G(t,s)f(s,z_s)\,ds + \int_{t_0}^{t} G(t,s)f(s,z_s)\,ds \\
&= U(t,t_0)\left[U(t_0+T,t_0)z^b(t_0) + \int_{t_0-T}^{t_0} G(t_0,s)f(s,z_s)\,ds\right] + \int_{t_0}^{t} G(t,s)f(s,z_s)\,ds \\
&= U(t,t_0)z_0 + \int_{t_0}^{t} G(t,s)f(s,z_s)\,ds,
\end{aligned}$$

which implies that

$$z(t) = U(t,t_0)z_0 + \int_{t_0}^{t} G(t,s)f(s,z_s)\,ds,$$

where

$$z_0 = U(t_0+T,t_0)z^b(t_0) + \int_{t_0-T}^{t_0} G(t_0,s)f(s,z_s)\,ds.$$

Therefore,

$$z(t) = z^b(t+T)$$

is a solution of (1) in the ball $\mathcal{B}^b_\rho(0)$. Hence by the uniqueness of the fixed point in this ball we get that

$$z^b(t) = z^b(t+T), \quad t > t_p.$$

□

Now, we shall prove that the bounded solution given by Theorem 1 and Theorem 2, under some conditions, is also almost periodic.

Let us assume the following hypotheses:
H8) $J_k = g = P = 0$ and the initial function $\phi \in \mathcal{PW}$ is almost periodic.

H9) $A(t) = A$ and $f : \mathbb{R} \times \mathcal{PW} \to \mathbb{R}^n$ is almost periodic in the first variable, uniformly in $\phi \in \mathcal{PW}$, and globally Lipschitz in ϕ.
We recall the following definition and a Theorem from (Toka, 2017).

Definition 1. *A jointly continuous function $f : \mathbb{R} \times \mathcal{PW} \to \mathbb{R}^n$ is almost periodic uniformly in $\phi \in S \subset \mathcal{PW}$, where S is a bounded set, if for any $\epsilon > 0$ there exists $\ell(\epsilon) > 0$ such that for any interval of the form $(\alpha, \alpha+\ell(\epsilon))$ contains η with the property*

$$\| f(t+\eta,\phi) - f(t,\phi) \| < \varepsilon, \quad \forall t \in \mathbb{R}, \quad \phi \in S.$$

Theorem 3. *Let $f : \mathbb{R} \times \mathcal{PW} \to \mathbb{R}^n$ be almost periodic in $t \in \mathbb{R}$, uniformly in $\phi \in S \subset \mathcal{PW}$, where S is bounded. Suppose that f is globally Lipschitz in $\phi \in \mathcal{PW}$. If $\zeta : \mathbb{R} \to \mathcal{PW}$ is almost periodic, the function*

$$\Gamma : \mathbb{R} \times \mathcal{PW} \to \mathbb{R}^n, \quad \textit{defined by} \quad \Gamma(t) = f(t,\zeta(t)),$$

is almost periodic.

Proposition 1. *Let $z \in \mathcal{PW}_b$ be an almost periodic function. Then, the function*

$$\begin{array}{rccl} \pi : & \mathbb{R} & \longrightarrow & \mathcal{PW} \\ & t & \longmapsto & \pi(t) = z_t, \end{array}$$

is almost periodic.

Proof. Since z is almost periodic, then for every $\epsilon > 0$ there exists $\ell(\epsilon) > 0$ such that any interval $(\alpha, \alpha+\ell(\epsilon))$ contains η such that

$$\| z(t+\eta) - z(t) \| < \varepsilon, \quad \forall t \in \mathbb{R}.$$

Hence

$$\| z(t+\eta+s) - z(t+s) \| < \varepsilon, \quad \forall t,s \in \mathbb{R}.$$

So,

$$\| \pi(t+\eta) - \pi(t) \| = sup_{S \in \mathbb{R}-} \| z_{t+\eta}(s) - z_t(s) \| < \varepsilon, \quad \forall t \in \mathbb{R}.$$

In consequence π is almost periodic. □

Now, for a function $\xi \in \mathcal{PW}_b$, we consider the set

$$H(\xi) = \overline{\{\xi_t : t \in \mathbb{R}\}},$$

the closure in the uniform convergence topology, it is called the Hull of ξ, and it is well known

((Toka, 2017)) that: ξ is almost periodic if, and only if, $H(\xi)$ is compact in the uniform convergence topology.

Also, the following statement holds:
For $\rho > 0$, the set

$$A_\rho = \{z \in \mathcal{B}^b_\rho : z \quad \text{almost periodic}\}$$

is closed.

Theorem 4. *Under the hypotheses* ***H8***)*-****H9***)*, the bounded solution z^b given by Theorems 1 and 2 is also almost periodic.*

Proof. In this case the bounded solution z^b can be written as follows

$$z^b(t) = \int_{-\infty}^{t} e^{A(t-s)} f(s, z_s)\, ds, \quad t \geqslant 0$$
$$z^b(s) = \phi(s), \quad s \in \mathbb{R}_-.$$

Now, consider the operator $\mathcal{K} : A_\rho \to \mathcal{B}^b_\rho$ given by (7). From proposition 1, we have that

$$\xi(t) = f(t, z_t), \quad z \in A_\rho$$

is almost periodic. On the other hand,

$$(\mathcal{K}z)(t) = \int_{-\infty}^{t} e^{A(t-s)} \xi(s) ds.$$

Next, we will show that $H(\mathcal{K}z)$ is compact in the uniform convergence topology. In fact, consider a

sequence $\{(\mathcal{K}z)_{\eta_n}\}$ in $H(\mathcal{K}z)$, where $(\mathcal{K}z)_{\eta_n}(t) = (\mathcal{K}z)(t+\eta_n)$. Since ξ is almost periodic, there exists a convergent sub-sequence $\{h_{\eta_{n_j}}\}$. Now, we have that

$$\begin{aligned}(\mathcal{K}z)_{\eta_{n_j}}(t) &= (\mathcal{K}z)\left(t+\eta_{n_j}\right) \\ &= \int_{-\infty}^{t+\eta_{n_j}} e^{A(t+\eta_{n_j}-s)} \xi(s) ds \\ &= \int_{-\infty}^{t} e^{A(t-s)} \xi(s+\eta_{n_j}) ds.\end{aligned}$$

Then,

$$\| (\mathcal{K}z)_{\eta_{n_j}}(t) - (\mathcal{K})_{\eta_{n_j}}(t) \| \leqslant \frac{M}{\beta} \| \xi_{\eta_{n_j}} - \xi_{\eta_{n_i}} \|_b .$$

Thus, $\{(\mathcal{K}z)_{\eta_{n_j}}\}$ is a Cauchy sequence in $\mathcal{PW}_b$, which implies that $\{(\mathcal{K}z)_{\eta_{n_j}}\}$ converges. Hence

$H(\mathcal{K}z)$ is compact, and $(\mathcal{K}z)$ is almost periodic function. so $\mathcal{K}(A_\rho) \subset A_\rho$. Therefore, the only fixed point of $\mathcal{K}$ on $\mathcal{B}^b_\rho$ is in A_ρ. Hence, $z^b(\cdot)$ is almost periodic. □

5 Conclusion and Final Remark

In this work, we prove the existence of bounded solutions for retarded equations with infinite delay, impulses, and non-local conditions. This is achieved assuming that the associated linear system has an exponential dichotomy and applying Banach's fixed point theorem. Then, under certain conditions, we prove that this bounded solution is stable; next, under the additional conditions, we prove that this bounded solution is periodic after the last time impulse t_p; in the same way, under certain conditions, we prove that this bounded solution is almost periodic. We believe that these results can be extended to evolution equations in infinite-dimensional Banach spaces; in fact, this constitutes our next research work in this direction.

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References

Abbas, S., Al-Arifi, N., Benchohra, M. & Graef, J. (2020). Periodic mild solutions of infinite delay evolution equations with non-instantaneous impulses. *Journal of Nonlinear Functional Analysis*. Article ID7.

Ayala, M., Leiva, H. & Tallana, D. (2020). Existence of solutions for retarded equations with infinite delay, impulses, and nonlocal conditions. submited for possible publication 2020.

Leiva, H. (1999a). Existence of bounded solutions of a second order system with dissipation. *J. Math. Anal. Appl.*, *237*, 288–301.

Leiva, H. (1999b). Existence of bounded solutions of a second order system with dissipation. *J. Math. Analysis and Applications*, pp. 288–302.

Leiva, H. (2000). Existence of bounded solutions of a second order evolution equation and applications. *J. Math. Phys.*, *41*.

Leiva, H. & Sequera, I. (2003). Existence and stability of bounded solutions for a system of parabolic equations. *J. Math. Analysis and Applications*, *279*, 495–507.

Leiva, H. & Sivoli, Z. (2003). Existence, stability and smoothness of a bounded solutions for a nonlinear time varying thermoelastic plate equations. *J. Math. Analysis and Applications*, *285*, 191–211.

Leiva, H. & Sivoli, Z. (2018). Existence, stability and smoothness of bounded solutions for an impulsive semilinear system of parabolic equations. *Afrika Matematika*, *24*, 1225–1235.

Liu, J. (2000). Periodic solutions of infinite delay evolution equations. *Journal of mathematical analysis and applications, Elsevier*, pp. 627–644.

Liu, J., Naito, T. & Minh, N. (2006). Periodic solutions of infinite delay evolution equations. *Journal of mathematical analysis and applications, Elsevier*, *286*, 705–712.

Toka, D. (2017). Well posedness for some damped elastic systems in bancha spaces. *Appl. Math Letter*, *71*.